

OBSTRUCTIONS TO COMBINATORIAL FORMULAS FOR PLETHYSM

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ABSTRACT. Motivated by questions of Mulmuley and Stanley we investigate quasi-polynomials arising in formulas for plethysm. We demonstrate, on the examples of $S^3(S^k)$ and $S^k(S^3)$, that these need not be counting functions of inhomogeneous polytopes of dimension equal to the degree of the quasi-polynomial. It follows that these functions are not, in general, counting functions of lattice points in any scaled convex bodies, even when restricted to single rays.

Many problems in representation theory have combinatorial solutions. A well-known, important example is the Littlewood–Richardson rule which gives the multiplicities of isotypic components in the tensor product of two irreducible $GL(n)$ representations [FH91]. The solution is combinatorial in the sense that the answer is given by the number of lattice points in explicit rational convex polyhedra, i.e. the multiplicities are equal to values of an Ehrhart quasi-polynomial. Berenstein and Zelevinsky provided another interpretation of the Littlewood–Richardson rule [BZ92], isomorphic, however, on the level of polytopes [PV05]. The study of different polyhedral structures turned out to be very useful. Knutson and Tao [KT99] showed that the honeycomb polytopes (in spirit similar to [BZ92]) are nonempty if and only if they contain a lattice point. This was the crucial last step in the solution to the Horn problem [Hor62] which goes back to Weyl’s work on eigenvalues of partial differential equations [Wey12].

In this paper we study the much more complicated plethysm coefficients $m_\lambda^{d,k}$ defined by

$$S^d(S^k W) = \bigoplus_{\lambda} m_\lambda^{d,k} S^\lambda W,$$

where the sum is over partitions λ of dk .

The computation of the multiplicity functions in plethysms can be seen as an operation on Schur polynomials [Mac98]. Viewed like this, it is surprising that when the plethysm is written in terms of other Schur polynomials, the coefficients are always nonnegative. Of course this must be true, because the coefficients are multiplicities of irreducible representations, but it would be desirable to have a combinatorial explanation for nonnegativity. In [Sta00, Problem 9] Richard Stanley asked for a *positive* combinatorial method to compute plethysm coefficients. A connection between plethysm and lattice point counting was shown at least in [KM15, Col15, CDW12]. These connections are not direct in the sense that plethysm coefficients are not seen to equal counts, but always involve some opaque arithmetics.

The functions $f(s) = m_{s\lambda}^{d,sk}$ and $g(s) = m_{s\lambda}^{sd,k}$ share many properties with Ehrhart functions of rational polytopes:

- Both f and g are nonnegative and $f(0) = g(0) = 1$.
- Both f and g are quasi-polynomials. This deep fact follows from [MS99, Corollary 2.12]. See [KM15, Remark 3.12].

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- For regular λ , the leading term of f is proportional to the Ehrhart function of a rational polytope corresponding to the Littlewood–Richardson rule [KM15, Section 4].

Example 1. The multiplicity of $\lambda = s(2k - 1, 1)$ inside $S^2(S^{sk})$ equals 0 when s is odd and 1 when s even. In particular it is equal to the number of integral points in the (zero-dimensional) polytope $\{s/2\}$. The multiplicity of $s(2k, k)$ inside $S^3(S^{sk})$ is equal to the number of integral points in the (one-dimensional) polytope $s[1/3, 1/2]$ or $s[1/2, 2/3]$.

Ketan Mulmuley posed several conjectures concerning the behavior of plethysm and Kronecker coefficients in relation to geometric complexity theory [MS01, Mul09].

Question 2. Fix positive integers k, d , and λ a partition of kd . Are the quasi-polynomials $s \mapsto m_{s\lambda}^{d,sk}$ and $s \mapsto m_{s\lambda}^{sd,k}$ Ehrhart functions of rational polytopes?

The question about $m_{s\lambda}^{sd,k}$ comes directly from [Mul09, Hypothesis 1.6.4]. Remark 5 gives an easy negative answer for both functions. Our main goal, however, is a generalized version of these questions for which we need some additional terminology. Following [Mul11, Section 5.1] we define a *shifted* or *inhomogeneous rational polytope* as a system of inequalities

$$P = \{x \in \mathbb{Q}^m : Ax \leq b + c\},$$

where b, c are arbitrary rational vectors and A is a rational matrix. Splitting the right hand side as $b + c$ is motivated by the definition of the dilations of P as

$$P(s) = \{x \in \mathbb{Q}^m : Ax \leq sb + c\}.$$

An *asymptotic Ehrhart quasi-polynomial* is a counting function of the form $s \mapsto \#(P(s) \cap \mathbb{Z}^m)$. The *dimension* of a shifted rational polytope P is by definition the dimension of $P(s)$ for large s (for small s the polytope can be empty). Contrary to the case of Ehrhart quasi-polynomials, an asymptotic Ehrhart quasi-polynomial does not need to be a quasi-polynomial, although it is for large arguments. Moreover, a quasi-polynomial may be an asymptotic Ehrhart quasi-polynomial but not an Ehrhart quasi-polynomial. Asymptotic Ehrhart quasi-polynomials do not have to satisfy Ehrhart reciprocity. Further, the dimension of the shifted rational polytope P may be strictly greater than the degree of the associated asymptotic Ehrhart quasi-polynomial. See also [Sta82] and [Sta96, Chapter I].

In [Mul11, Hypothesis 5.3] it is conjectured that the multiplicity of $s\pi$ in $S^{s\lambda}(S^\mu)$ is an asymptotic Ehrhart quasi-polynomial with additional complexity-theoretic properties. In the present paper we make progress towards a negative answer of the following weakened version.

Question 3. Is $m_{s\lambda}^{sd,k}$ an asymptotic Ehrhart quasi-polynomial?

Our main result, Theorem 4, implies a negative answer to both cases in Question 2 and strong restrictions on a positive solution to Question 3. We show that $m_{s\lambda}^{sd,k}$ need not be an asymptotic Ehrhart quasi-polynomial of a polytope of dimension equal to the degree of its growth. At the moment we are not able either to exclude or confirm that $m_{s\lambda}^{sd,k}$ is an asymptotic Ehrhart quasi-polynomial for an inhomogeneous polytope of dimension strictly larger than its degree.

In previous work the authors gave a formula for plethysm coefficients, which is a sum of Ehrhart functions of various polytopes with (positive and negative) coefficients [KM15]. This allows to gather experimental data on the questions for many rays. A-posteriori, the specific plethysm in Theorem 4 can also be confirmed using well-known formulas for $S^3(S^k)$. General formulas for $S^k(S^3)$ are unknown, though. Our methods are inspired by [KW15, BOR09].

Theorem 4. *The multiplicity functions of $S^{s(7,5,0)}$ in $S^3(S^{4s})$ and $S^{4s}(S^3)$ equal*

$$\phi : s \mapsto \frac{s + r(s)}{3}$$

where $r(s)$ has period 6 and takes the values $3, -1, 1, 0, 2, -2$ on respectively the integers $0, \dots, 5$. There do not exist rational vectors $a, b, c \in \mathbb{Q}^n$ (of arbitrary length n) such that ϕ equals the counting function of a one-dimensional inhomogeneous polytope

$$P = \{x \in \mathbb{Q} : ax \leq b + c\}.$$

In particular, ϕ is not an Ehrhart function of any rational polyhedron.

Remark 5. Before giving a general proof, it is easy to see that ϕ cannot be an Ehrhart function: it violates Ehrhart–Macdonald reciprocity [Ehr67, Mac71]. The value $f(-s)$ of an Ehrhart quasi-polynomial f of a rational polytope P at a negative integer argument counts (up to a global sign) the number of interior lattice points in sP . In particular, $|f(-s)| \leq f(s)$. However,

$$-\phi(-1) = 1 > 0 = \phi(1).$$

Interestingly, the jumps in values that make Ehrhart reciprocity fail are also the crucial ingredient for our proof of non-representability by a one-dimensional inhomogeneous polytope.

Proof. The equality of the multiplicities of the given ray in both plethysms is a consequence of Hermite reciprocity [Her54], [FH91, Exercise 6.18]. There are now various ways to determine the formula from the interpretation as a multiplicity in $S^3(S^{4s})$. One is to simply evaluate the explicit formula from [KM15] along a ray. Another way is to observe that the function must exhibit linear growth and that its period is at most six [CGR84, MM15]. With this information and some values the function can be interpolated.

Let P be an inhomogeneous line segment as in the statement. Then the s -th dilation of P can be written as

$$P(s) = \{x \in \mathbb{Q} : \max_i(sb_i + c_i) \leq x \leq \min_i(sb'_i + c'_i)\}.$$

Asymptotically, $P(s)$ becomes an interval of length $\frac{s}{3}$. This means that there exists an s_0 such that for all $s > s_0$

$$(1) \quad P(s) = \{x \in \mathbb{Q} : sb + c \leq x \leq s(b + \frac{1}{3}) + c'\} =: Q(s),$$

for some $b, c, c' \in \mathbb{Q}$. Making s_0 even larger it can be assumed that s_0b are integers and that s_0 is divisible by 6 (the period of ϕ). Since $\#Q(s)$ and $\#P(s)$ have the same linear term and constants that are s_0 -periodic (by the divisibility assumptions) they agree for all s .

The proof of nonexistence of the family $Q(s)$ such that $\phi(s) = \#Q(s)$ is by examination of constraints on the coefficients b, c, c' , coming from the known values of the counting function ϕ . Without loss of generality we have

$$(2) \quad 0 < c \leq 1,$$

$$(3) \quad 0 \leq b < 1.$$

Here (2) can be assumed since we can shift Q by integral values without changing the counting function, and (3) can be assumed because changing b by an integer also only shifts Q by an integral value. Specializing (1) to $s = 0$ together with $\phi(0) = 0$ gives

$$(4) \quad 1 \leq c' < 2,$$

because the single integer in $Q(0)$ must be 1. With $s = 1$ we get two new general bounds

$$(5) \quad 1 < b + c,$$

$$(6) \quad b + c' \leq \frac{5}{3}.$$

We have $\phi(5) = 1$ and $\phi(4) = 2$. The possible integers in $Q(5)$ using all constraints so far are 2, 3, 4, 5, 6. The possible pairs of integers in $Q(4)$ are (2, 3), ..., (5, 6). Assume $Q(5)$ contains 6, then $5 < 5b + c$ and $4 < 4b + c$ from the lower bound in (1) and $6 \leq 4b + \frac{4}{3} + c'$ from the upper bound. This implies $\frac{14}{3} \leq 4b + c' \leq 3b + \frac{5}{3}$ (by (6)) and thus $3 \leq 3b$, in contradiction to (3). This excludes the possibilities $6 \in Q(5)$ and $5, 6 \in Q(4)$. Since $6 \notin Q(5)$, we get $6 > 5b + \frac{5}{3} + c'$ from the upper bound on $Q(5)$ and then using (4) a new global bound

$$(7) \quad b < \frac{2}{3}.$$

Now, $2 \in Q(5)$ is impossible, since the two integers in $Q(4)$ have to be less than or equal to the integer in $Q(5)$. We exclude that 4, 5 are the integers in $Q(4)$. Indeed, the upper bound on $Q(4)$ is $4b + \frac{4}{3} + c' \stackrel{(6)}{\leq} 3b + 3 \stackrel{(7)}{<} 5$. This in turn implies that $5 \notin Q(5)$ since $4b + c \leq 3$ implies $5b + c \leq 4$. As above, we now get $5 > 5b + \frac{5}{3} + c'$. With (4) we find

$$(8) \quad b < \frac{7}{15}.$$

We claim that if $b \leq \frac{2}{5}$, then $b < \frac{4}{15}$. Indeed, in this case $5b + c \leq 3$ which since $\phi(5) = 1$ implies $5b + \frac{5}{3} + c' < 4$, from which $b < \frac{4}{15}$ follows by (4). After this claim there are two possible intervals for b , namely $b \in [\frac{3}{15}, \frac{4}{15})$ or $b \in (\frac{6}{15}, \frac{7}{15}]$. Consider $Q(6)$ which should contain 3 integers. The possible triples are (3, 4, 5) and (4, 5, 6). In both cases from the upper bound on $Q(6)$ we have $6b + 2 + c' \geq 5$. Using (6) we get $b \geq \frac{4}{15}$, hence $b \in (\frac{6}{15}, \frac{7}{15}]$. From the lower bound on $Q(6)$ we find $6b + c > 5b + 1 > 3$ which excludes (3, 4, 5) from being the triple in $Q(6)$. Finally, if 4, 5, 6 are in $Q(6)$, then from the upper bound on $Q(6)$ we now get $6b + 2 + c' \geq 6$ and thus, with (6), $b > \frac{7}{15}$. With no possible values for b left, the function ϕ cannot be a counting function for an interval as in (1). \square

Remark 6. We have confirmed that all examples with two rows and fewer boxes are in fact Ehrhart functions. In this sense, our example is minimal.

Remark 7. A different way to describe an inhomogeneous polytope is by linear integral equations and nonnegativity. In this setting Stanley gave a general reciprocity theorem which relates the Hilbert series (expanded at ∞) of the module of positive integral solutions of linear Diophantine equations to the Hilbert series of negative solutions to the same equations. In [Sta82, Theorem 4.2] there are, however, combinatorially defined correction terms.

Remark 8. While this is not visible from the representation in [KM15], a general theorem of Meinrenken and Sjamaar [MS99] implies that the chambers of plethysm quasi-polynomials are polyhedral cones (see [KM15, Remark 3.12], [PV15]). One may thus ask the stronger question if they are Ehrhart quasi-polynomials in general. Of course, Theorem 4 gives a negative answer to this much stronger property too.

Lattice point counting in polytopes and Ehrhart functions are arguably the most natural combinatorial explanations one may hope for here, but they are not the only ones. Our theorem also excludes other possibilities.

Corollary 9. *The plethysm coefficients are not positive combinations of counting functions of lattice points in integral scalings of any convex regions.*

Proof. One-dimensional convex sets are polyhedra and the counting function in Theorem 4 would need to specialize to an Ehrhart function. \square

The plethysm counting function in Theorem 4 cannot be written as a positive combination of Ehrhart quasi-polynomials as these also have to satisfy Ehrhart reciprocity. However, it can be written as a sum of an asymptotic Ehrhart quasi-polynomial and an honest Ehrhart quasi-polynomial as in the following proposition (which is very easy to confirm).

Proposition 10. *Fix a small rational $\varepsilon > 0$. Let $P(s) = s[\varepsilon, \frac{1}{3} + \varepsilon]$ and $Q(s) = s\{\frac{1}{2}\}$. Then $\phi(s) = \#P(s) + \#Q(s)$.*

Remark 11. An interesting open question is the following. Suppose we are given a piecewise quasi-polynomial whose chambers are cones. Is there a finite computational test if the quasi-polynomial is an (asymptotic) Ehrhart function? Assume it is not an Ehrhart function globally, but is an Ehrhart function on each individual ray. Is there a finite number of rays such that finding polytopes for each of the finitely many rays proves the Ehrhart nature for all rays? Is it some multi-dimensional generalization of Apéry sets known from numerical semigroups?

Remark 12. According to [KM15, Remark 3.8] the function ϕ in Theorem 4 also counts, for example, the multiplicity of $\lambda = s(7 + 2t, 5 + 2t, 2t)$ in $S^3(S^{s(5+2t)})$ for any positive t . Therefore counterexamples also exist for strictly interior rays.

Remark 13. Recently, there has been a lot of interest in Kronecker coefficients [Man14, BOR09, SS16, BV15]. These are different from plethysm, but also hard to compute and important in geometric complexity theory. It is known that on rays they are given by quasi-polynomials [Man15, Theorem 1] that do not have to be Ehrhart functions [KW15]. However, the problem is open for specific rays that are important in GCT.

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